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that a better insight into the significance of mathematics will prevent or nullify mistaken attacks on the subject as one of little public worth.

Such to my mind should be one function, if not the chief function, of this Association: the regeneration of a significant mathematics, the encouragement of workers in applied mathematics, and the effort to obtain recognition of the true public worth of mathematics in every phase.

THE LINEAR FUNCTION AND THE LINE.

By JOSEF NYBERG, Chicago, Ill.

In a previous paper[1] I showed how the work in college algebra, trigonometry, and analytics may be unified by treating the contents of these subjects under the general theory of functions. In a second paper[2] I then showed how the notion of function should be introduced so that the student may see that coordinates constitute only the language of mathematics, the subject itself dealing with the properties of functions and relations between variables. The present paper begins the study of the line, and closes with some illustrations of how the new point of view eliminates certain difficulties.

I begin the work by stating that the rate of increase of a function is one of its most important properties, and that the simplest function is that which has a known constant rate of increase. For such a function we may define the rate as the ratio of the change in the value of the function to an increase in the value of the variable. We can measure this rate by the change in the function per unit increase in the variable. This idea is impressed by writing some data on the board, as

$$\left\{\begin{array}{lrrrrr} x & -1 & 0 & 1 & 2 & 3 \\ y & 2 & 5 & 8 & 11 & 14, \end{array}\right. \qquad \left\{\begin{array}{lrrrr} x & -1 & 1 & 3 & 5 \\ y & 4 & 9 & 14 & 19, \end{array}\right.$$

and asking such progressive questions as: What is the change in x? in y? What is the rate of increase? What is the value of y for $x = 0$? How can the value of y be computed for a value of x not given in the data? When the student has seen that the general relation is $y = y_0 + mx$ the more formal work may be presented in the following order.

I. Assuming that the function has a constant rate of increase m, and that several corresponding values are known, we prove that the graph of the function is a line; and conversely, a line is the graph of a function with a constant rate of increase. The proof naturally hinges on the similarity of certain triangles; and as these triangles will be similar only when m is constant, the student will associate a line with a uniform rate. I prefer the form $y = y_0 + mx$ to the form $y = mx + b$, because the physicists and engineers are accustomed to such relations as $v = v_0 + at$, $s = s_0 + vt$, $p = p_0 + (\rho g)h$ and because $y = y_0 + mx$

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[1] "The Unification of Freshman Mathematics," in this MONTHLY, April, 1916, page 101.

[2] "The Presentation of the Notion of Function," in this MONTHLY, September, 1917, p. 309.

is the only statement of how the student actually does compute any value of y in a numerical case: he adds to y_0, a reference value, the amount mx, which is proportional to the increase in x from 0, another reference value. Even if we are dealing with an abstract relation, b would not be as good a symbol as y_0 because it is entirely lacking in connotation.

II. As a second step we show formally that every linear relation $Ax + By + C = 0$ $(B \neq 0)$ can be written as $y = y_0 + mx$ and that its graph is a line.

III. If the value of y_0 is not known, but the rate of increase and a pair of corresponding values are known, we do not need to compute y_0 explicitly, as the student did in the preliminary exercises, but can show that the relation is $y = y_1 + m(x - x_1)$. Further, if the rate of increase is unknown the student will see that the function is undetermined unless we know several corresponding values from which the rate can be found as

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The work thus emphasizes that a pair of values and a rate determine a function, and does not particularly emphasize such an equation as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Even the discussion of an abstract line can be based on the form

$$y = y_1 + m(x - x_1)$$

for it can be used to derive the "two intercept" form and the "perpendicular" form, wherein a point $(p \cos \alpha, p \sin \alpha)$ and a slope $(-\cot \alpha)$ are known. I do not consider here the other problems dealing with families of lines or locus problems, as I am concerned only with the introduction to the subject.

There are two points in this method to which I call attention. The first is that all the beginning work is grouped around a central idea—the study of a function determined by a value and a rate of increase. This idea unifies the separate problems. Examining the corresponding treatment of lines in any textbook, we note the great diversity even in a single book in the modes of attack on these problems. Such problems as the line through two points, the point-slope form, the perpendicular form, the determination of the slope from the equation, the condition that lines be parallel or perpendicular, all these appear as *distinct* problems, completed only when a formula is found for writing the answers to analogous problems. Thus, the practice of using an equation like

$$\frac{y - y'}{y' - y''} = \frac{x - x'}{x' - x''},$$

found in several books, is especially pernicious in that it has absolutely nothing in common with other forms, and because the simpler concepts on which the concept of a line is built are wholly missing. Again, the use of projections in finding the perpendicular form of a line is undesirable because it seems to the student to be only another trick of the omniscient teacher. By the present method, however, finding the perpendicular form of a line involves no trick, for the various forms are based on one as fundamental. Similarly the equation of a line through two points is easily derived by the student when he sees the data to be only a bit of information for determining the slope. Although we do not ordinarily ask the student to discover such relations, the fact that he can is a test of his understanding and a proof that he is thinking, not memorizing. When the problems do not center around an idea the student is bound to resort to memorizing.

Secondly, I wish to remark on the pedagogical correctness of the procedure. We know that the permanence of an idea depends on the interest with which it was first aroused and the attention which that interest stimulated. Psychology thus requires that we begin with as concrete a problem as possible, and that we base it on the student's previous knowledge. The student is familiar with the notion of rate. Every day he has determined the position of some object, a train, a car, or a person he wishes to meet, by knowing the rate of motion and the position at some moment. Further, concreteness in its broad sense means not only that we begin with a study of $3x + 4y + 5 = 0$ instead of $ax + by + c = 0$, but that we begin with some very definite question about the line $3x + 4y + 5 = 0$, and that this question be fundamental, not accessory. Certainly the slope of a line is more significant than its intercepts, or the points that it may pass through, or the appearance of its graph.

As an illustration of the usefulness of the phrase "relation between variables," consider the problem of finding the coordinates of a point P_0 dividing a segment $P'P''$ in a given ratio and leading to the equation

$$x_0 = \frac{k_1 x'' + k_2 x'}{k_1 + k_2}.$$

To the student this is a formula for computing x_0 when P' and P'' are given, and he experiences no difficulty with it except in two cases: (1) when $P'P''$ is divided externally, and (2) when P_0, P', and P'' are given and the ratio is to be found. On the other hand, given two points (2, 3) and (12, 8), and asked to find a point three fifths of the way between them, the student will compute directly that the desired point is $2 + \frac{3}{5}(12 - 2), 3 + \frac{3}{5}(8 - 3)$. Hence, calling the points P_1, P_2, P_3 we have $x_2 = x_1 + \lambda(x_3 - x_1)$ where λ is a certain ratio. The student should never hear that this is a formula for computing x_2 but should be taught that it is a statement of a relation between four quantities x_1, x_2, x_3, λ. *Any three* of the quantities being known, this relation enables us to determine the *fourth.* The notation $P_0P'P''$ attracts undue attention to P_0, while the notation $P_1P_2P_3$ puts all the points on an equal footing giving merely the relation between them. This point of view will eliminate the second difficulty mentioned above. The first difficulty is treated by always lettering the points from left to right

$P_1P_2P_3$ (irrespective of which is the unknown), determining λ from the data (if it is not given explicitly), and then applying the relation. I have found that the complete elimination of the word "formula," in terms of which the student likes to think, and the substitution of the idea that the equation is a statement of a relation between various quantities is very helpful in all such problems.

A similar situation is met when dealing with the angle between two lines. The student can use the formula

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

to find $\tan\theta$, but the problem of finding a line making a given angle with a given line is more difficult. This trouble, like the previous one, is avoided by impressing the fact that $(1 + m_1m_2)\tan\theta = m_1 - m_2$ is a relation between three quantities, m_1, m_2, and $\tan\theta$, any two of which being known the relation enables us to determine the third. It is not a mere change of terminology which is used for removing the difficulty; we are giving the student a better conception of the word *equation.*

THE GRAPH OF $F(X)$ FOR COMPLEX NUMBERS.

By A. F. FRUMVELLER, Marquette University.

Teachers of analytic geometry are often asked why the complex roots of $f(x)$ cannot be diagrammed. The answer usually given, that real points of the graph lie in the plane of the blackboard, while complex points are in front of this plane or behind it, is true enough; but it leaves the student unsatisfied. He is not able to carry out the idea for himself; and the teacher may not have thought of working it out, nor can he lay his hand on any systematic treatment of the matter. Yet the topic is important, not to say interesting, since several lines of thought here converge: a rather full development with ample illustrations may consequently be welcomed. The purposes of a mathematical club, or undergraduate seminar, would be beautifully served by a handling of graphs along the following lines. Care has been taken with the notation: x invariably stands for $u + iv$; y stands for $U + iV$; complex constants are designated by Greek letters; thus, $\alpha = a + iA$, $\rho = r + iR$, etc. We restrict $f(x)$ to algebraic types merely for the sake of uniformity in the presentation.

The literature on this subject is so slight that the writer is able to mention only three sources to which the student may go for further information; these are:

1. Schultze's "Graphic Algebra" (The MacMillan Company, 80 pp.): methods are given for the construction of the complex roots of quadratics, cubics, and quartics, the purpose being merely to find these roots, and not to exhibit the complete graph.

2. Hamilton and Kettle's "Graphs and Imaginaries" (Longmans, Green and Company, 40 pp.): the quadratic alone is treated. Complex roots, imaginary